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Bayesian Survival Analysis in Reliability for Complex System with a Cure Fraction

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Abstract: In traditional methods for reliability analysis, one complex system is often considered as being composed by some subsystems in series. Usually, the failure of any subsystem would be supposed to lead to the failure of the entire system. However, some subsystems' lifetimes are long enough and even never fail during the life cycle of the entire system. Moreover, such subsystems' lifetimes will not be influenced equally under different circumstances. In practice, such interferences will affect the model's accuracy, but it is seldom considered in traditional analysis. To address these shortcomings, this paper presents a new approach to do reliability analysis for complex systems. Here a certain fraction of the subsystems is defined as a "cure fraction" under the consideration that such subsystems' lifetimes are long enough and even never fail during the life cycle of the entire system. By introducing environmental covariates and the joint power prior, the proposed model is developed within the Bayesian survival analysis framework, and thus the problem for censored (or truncated) data in reliability tests can be resolved. In addition, a Markov chain Monte Carlo computational scheme is implemented and a numeric example is discussed to demonstrate the proposed model.

Keywords: *Bayesian analysis, survival analysis, reliability, Markov chain Monte Carlo, cure rate model, power prior*

1. Introduction

In practice, under varying circumstances, failures of a portion of subsystems (or units) will lead to the failure of a complex system (or module). Meanwhile, the lifetimes of other portion of subsystems will far exceed the lifetime of the system itself. In other words, the latter will not lead to the failure of the system. To exemplify, compared with the whole life cycle of a compressor, the lifetime of some screws' will far exceed the lifetime of the compressor. However, the failures of the gears may directly lead to failure of the compressor. This aspect is often neglected in traditional reliability analysis and is especially important when constructing a regression model such as the proportional hazards model. In the above example, a regression based model is vital since the lifetime for one kind of compressor could vary when used in different environment settings (different temperature, moisture, running skill, and the like). Hence, when considering environmental factors, the weights for all subsystems' influences subject to different environments could not be viewed as equal. When some subsystems never fail, this will influence the regression and affect the degrees of freedom.

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An additional factor that is commonly neglected is that the number of these subsystems can vary and should be treated as a random variable whose characteristics changes with different circumstances. Obviously, the accuracy of a model will improve by considering these aspects including interference for the covariates, which lays the foundation for us to utilize the model mentioned below in reliability analysis.

Cure rate models, which have been used to model time-to-event data, are obtained by survival models incorporating a cure fraction. Perhaps the most popular type of cure rate model is the mixture model discussed by Berkson and Gage [1] as in (1).

$$S_p(t) = (1-\pi)S^*(t) + \pi \tag{1}$$

Let $S_p(t)$ denote the survivor function for the entire population, let π be defined as the fraction of the populations which is "cured", and let $S^*(t)$ represent the survivor function for the non-cured group in the population. The model given in (1) is increasingly popular when analyzing data from cancer clinical trials, where a certain fraction π of the population is assumed to be "cured". This is commonly referred to as the *standard cure rate model*. Common models for $S^*(t)$ include the exponential and Weibull distributions. Clearly, as $\pi \to 1$, $S_p(t)$ tends to 1, whereas as $\pi \to 0$, $S_p(t)$ tends to $S^*(t)$. The standard cure rate model has been extensively discussed in the statistical literature, including Gray and Tsiatis [2], Kuk and Chen [3], Taylor [4], Sy and Taylor [5], Peng and Dear [6], Betensky and Schoenfeld [7]. Furthermore, the book by Maller and Zhou [8] provided an extensive discussion on frequentist inference for the standard cure rate model. However, due to some drawbacks of the standard cure rate model, such as the inability to introduce covariates in π and causing significant inconvenience when utilizes the Bayesian framework, using prior information, an alternative definition of the cure rate model has been proposed, and investigated by authors including Yakovlev and Tsodikov [9], Chen, Ibrahim and Sinha [10], among others. This is commonly referred to the *promotion time cure model*. The promotion time cure model is strongly motivated from biological considerations, and the Bayesian formulation of this model is given by Chen, Ibrahim and Sinha [10]-[13], which is also the foundation of the model in this article. Up-to-date research results can be found in Yin and Ibrahim [14], and Zeng, Yin and Ibrahim [15].

The rest of this article is organized as follows. First, we introduce the standard cure rate model, and then the Bayesian formulation of the model for reliability analysis in complex system. In our model, we view a certain fraction of subsystems, whose lifetimes far exceed the lifetime of the system, as a "cure fraction"; second, we propose a regression model that includes covariate structure for the cure fraction. This is achieved by introducing environment covariates. Subsequently, the joint power prior which incorporates "historical data" is proposed. In this way, we propose a new way to optimize the reliability of complex system. Markov chain Monte Carlo (MCMC) methods, based on Gibbs sampling, is used to infer properties of the parameters' posterior distribution. Finally, the end results for the model under random truncated conditions are presented; and the validity of the model is proved by the example.

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2. Cure Rate Model in Reliability

2.1 Standard Cure Rate Model

For the sake of convenience, complex systems are often considered to be composed in terms of some subsystems in series. This means that the failure of any subsystem will lead to the failure of the entire system. As shown in Fig.1, suppose that the complex system (or module) is composed by j ($j = 1, \cdots, N, \cdots N_0$, where N_0 may be unknown) subsystems (or units).

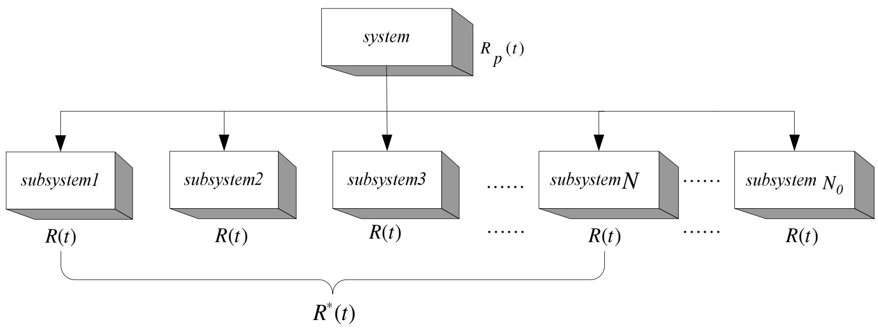

Figure 1: Structure for Complex System in Series

Let the lifetime of each subsystems is denoted by t_j , $j = 1, \cdots, N, \cdots N_0$. Suppose that, when running under different circumstances (different temperature, moisture, running skill, and the like), the failure of some subsystems (or units) will lead to the failure of the entire complex system (or module), whereas the lifetimes of another portion of subsystems far exceed the lifetime of the entire system, which is denoted by T . In other words, here the latter will not lead to the failure of the entire system. For example, compared with the whole life span of a compressor, some screws' lifetimes will far exceed the lifetime of the compressor. However, the failures of the gears may directly lead to the compressor's failure. Furthermore, the lifetime for one kind of compressor is different when used in different environments. When constructing a regression model (such as the proportional hazards model) to incorporate the covariates under different circumstances, those subsystems, whose lifetimes are long enough compared with the entire system, will rarely be influenced by the underlying circumstances. Let the random variable N （$N \le N_0$） denote the number of subsystems that lead to the failure of the system under different circumstances. Accordingly, the number of subsystems that will not lead to the failure of the system equals $N_0 - N$. In the following analysis we assume the lifetime of the entire system $T = \min(t_1, \cdots t_N)$, and T are independent of $t_{N+1}, \cdots, t_{N_0}$. For notation, let $R_p(t)$ represent the reliability function of the entire system, and let $R^*(t)$ denote the reliability determined by the N subsystems that lead to the failure of the system. Considering the standard cure rate model, we can get

$$R_p(t) = (1-\pi)R^*(t) + \pi \tag{2}$$

where $\pi = (N_0 - N)/N_0$ indicates the fraction of the subsystems that will not lead to the failure of the system. Obviously, $R_p(t) = 1$ as $\pi \to 1$ ($N \to 0$), which means that the lifetime of the system will not be influenced by the included subsystems). Also, $R_p(t) = R^*(t)$ as $\pi \to 0$ ($N \to N_0$), which indicates that nearly any subsystem will lead to the failure of the entire system). We note that, here (2) becomes the traditional lifetime model.

2.2 Alternative Model

As pointed out by Chen, Ibrahim and Sinha [10]-[13], although the standard cure rate model is attractive and widely used, it has two main drawbacks: (1) in the presence of covariates, a proportional hazards structure cannot be employed if the covariates are modeled through π via a binomial regression model. It hence lacks the desirable property of proportionality when employing covariate analysis; (2) Bayesian inference with the standard cure rate model essentially requires a proper prior, which is a limitation when the model is used in practice. Governed by these limitations, we resort to the promotion time cure model. The Bayesian formulation of this model can be found in Chen, Ibrahim and Sinha [10]-[13]. Based on this model, we discuss how the method can be used in reliability analysis of complex system.

From now on, we assume N is a Poisson distributed random variable with mean θ, denoted $N \sim Poi(\theta)$; let $t_k (k = 1,\cdots,N)$ represent the lifetime of the k *th* subsystem. We note here that failure here directly will lead to failure of the entire system. Given N, the random variables $t_k (k = 1,\cdots,N)$ are assumed to be independent and identically distributed (*i.i.d*) random variables with a common distribution $F(t) = 1 - R(t)$ that does not depend on N. The lifetime of the system $T = \min(t_k, 1 \le k \le N)$, and therefore, the reliability function for the entire system can then be given as

$$\begin{aligned}
R_p(t) &= P(T > t) \\
&= P(N = 0) + P(t_1 > t,\cdots,t_N > t / N \ge 1)P(N \ge 1) \\
&= \exp(-\theta) + \sum_{k=1}^{\infty} R(t)^k \cdot \frac{\theta^k}{k!}\exp(-\theta) \\
&= \exp(-\theta)\left[1 + \frac{\theta R(t)}{1!} + \cdots + \frac{[\theta R(t)]^k}{k!}\right] \\
&= \exp((-\theta) + \theta\, R(t)) \\
&= \exp(-\theta F(t))
\end{aligned} \tag{3}$$

We note that $R_p(t) = P(N = 0) = \exp(-\theta)$ as $t \to \infty$, where the cure fraction $\pi = \exp(-\theta)$; Chen [10] has pointed out that aside from biological motivation; the model in (3) is suitable for any type of survival data which include a surviving fraction. The probability density function and hazard function corresponding to (3) is given by:

$$f_p(t) = \frac{d[1 - R_p(t)]}{dt} = \theta \cdot f(t)\exp(-\theta F(t)) \tag{4}$$

$$h_p(t) = \frac{f_p(t)}{R_p(t)} = \theta \cdot f(t) \tag{5}$$

The cure rate model in (3) yields an attractive form for the hazard in (5). In particular, $h_p(t)$ is multiplicative in θ and $f(t)$, and thus it has a proportional structure when the covariates are modeled through θ. The proportional hazards property in (5) is also computationally attractive, as MCMC sampling methods are relatively easy to implement. For the N subsystems that will lead to the failure of the system, the reliability function, probability density function and hazard rate function are given by

$$R^*(t) = P(T > t \mid N \geq 1) = \frac{P(T > t) - P(N = 0)}{1 - P(N = 0)} = \frac{\exp(-\theta F(t)) - \exp(-\theta)}{1 - \exp(-\theta)} \tag{6}$$

$$f^*(t) = \frac{d[1 - R^*(t)]}{dt} = \theta \cdot f(t) \cdot \frac{\exp(-\theta F(t))}{1 - \exp(-\theta)} \tag{7}$$

$$h^*(t) = \frac{f^*(t)}{R^*(t)} = \theta \cdot f(t) \cdot \frac{\exp(-\theta F(t))}{\exp(-\theta F(t)) - \exp(-\theta)} \tag{8}$$

We can get (9) by incorporating (6) with (3), by that we can get the relationship between the model in (1) and (3).

$$R_p(t) = \exp(-\theta) + (1 - \exp(-\theta))\frac{\exp(-\theta F(t)) - \exp(-\theta)}{1 - \exp(-\theta)} \tag{9}$$

We note that, with $\pi = \exp(-\theta)$, $\pi \to 0$ and $R_p(t) = R^*(t)$ as $\theta \to \infty$; $\pi \to 1$ and $R_p(t) = 1$ as $\theta \to 0$.

2.3 Likelihood Function with Random Truncation

In reliability analysis, the lifetime data is usually "truncated" (or "censored") , which means the lifetimes are known for only a portion of the units under study, and the remainders of the lifetimes are known only to exceed certain values. The random truncated test can be described as follows: suppose the i th ($i = 1, \cdots, n$) unit has a life time T_i and truncated time L_i ; T_i and L_i are independent and their reliability functions are $R(t_i)$ and $G(t_i)$, with probability density functions $f(t_i)$ and $g(t_i)$, respectively. Suppose that only the lifetime $\mathbf{t} = (t_1, \cdots t_n)$ can be observed, where $t_i = \min\{T_i, L_i\}$. The ith individual will be considered lost (truncated or censored) if $T_i > L_i$. Denote the truncated indicators by υ_i, where $\upsilon_i = 1$ if $T_i \leq L_i$, and $\upsilon_i = 0$ if $T_i > L_i$. Then the likelihood function based on the theory of Bayesian survival is then

$$\begin{aligned} L(t) &\propto \prod_{i=1}^{n} [G(t_i) f(t_i)]^{\upsilon_i} [g(t_i) R(t_i)]^{1-\upsilon_i} \\ &\propto \prod_{i=1}^{n} [f(t_i)]^{\upsilon_i} R(t_i)^{1-\upsilon_i} \\ &\propto \prod_{i=1}^{n} [-\frac{d}{dt_i} P(T \geq t_i | N_i)]^{\upsilon_i} [P(T \geq t_i | N_i)]^{1-\upsilon_i} \end{aligned} \tag{10}$$

Now, suppose that we have n units in the test. Let N_i denote the number of subsystems that will lead to the failure of the system for the ith unit, and let t_{i,N_i} denote the lifetime of the N_i th subsystem for the ith unit. It should emphasize that, the N_i 's are not observed, but rather viewed as latent variables in the model formulation. Further, suppose the t_{i,N_i} s' are i.i.d and Weibull distributed with shape parameter α and scale parameter γ . The probability density function is then $f(\mathbf{t}) = \alpha\gamma\mathbf{t}^{\alpha-1}\exp(-\gamma\mathbf{t}^{\alpha})$. By letting $\lambda = \log(\gamma)$, then $f(\mathbf{t} \mid \alpha, \lambda) = \alpha \mathbf{t}^{\alpha-1} \exp(\lambda - \mathbf{t}^{\alpha} \exp(\lambda))$, denoted by $\mathbf{t} \sim W(\lambda, \alpha)$. Let $\mathbf{X}$ denote $n \times p$ vector of covariates for the i th unit, hence $\mathbf{X}$ represents the covariates studied in reliability trials, that may be the main environment factors that influence the life distribution, which we have to chose when the model being constructed. Meanwhile, let $\boldsymbol{\beta}$ be the $n \times p$ vector of the regression coefficients and denotes the degree of influences of covariates. Let the observed data for current study denoted by $D = (n, \mathbf{t}, \boldsymbol{\upsilon}, \mathbf{X})$. By incorporating it with (9), N_i ($N_i \sim P(\theta_i)$) as well as covariates $\theta_i = \exp(\mathbf{x}_i^{'} \beta)$, the likelihood function can be written as

$$\begin{aligned} L(\boldsymbol{\theta}, \alpha, \lambda \mid D) &\propto \prod_{i=1}^{n} \{[-\frac{d}{dt_i} P(T \geq t_i | \alpha, \lambda)]^{\upsilon_i} [P(T \geq t_i | \alpha, \lambda)]^{1-\upsilon_i} \times P(N_i | \theta_i)\} \\ &\propto \prod_{i=1}^{n} [-\frac{d}{dt_i} R(t_i | \alpha, \lambda)^{N_i}]^{\upsilon_i} [R(t_i | \alpha, \lambda)^{N_i}]^{1-\upsilon_i} \times \prod_{i=1}^{n} \frac{\theta_i^{N_i}}{N_i!} \exp(-\theta_i) \\ &\propto \prod_{i=1}^{n} R(t_i \mid \alpha, \lambda)^{N_i - \upsilon_i} (N_i f(t_i \mid \alpha, \lambda))^{\upsilon_i} \times \\ &\quad \exp[\sum_{i=1}^{n} (N_i \log(\theta_i) - \log(N_i!)) - n\theta_i] \\ &\propto \prod_{i=1}^{n} R(t_i \mid \alpha, \lambda)^{N_i - \upsilon_i} (N_i f(t_i \mid \alpha, \lambda))^{\upsilon_i} \times \\ &\quad \exp[\sum_{i=1}^{n} (N_i \mathbf{x}_i^{'} \boldsymbol{\beta} - \log(N_i!)) - n \exp(\mathbf{x}_i^{'} \boldsymbol{\beta})] \end{aligned} \tag{11}$$

where $f(t_i \mid \alpha, \lambda) = \alpha t_i^{\alpha-1} \exp(\lambda - t_i^{\alpha} \exp(\lambda))$. To express our final model, we hence need to infer the values of α, λ and β.

3. Bayesian Analysis based on MCMC

3.1 The Priors

Let $\pi(\cdot)$ denote the prior or posterior distributions for the parameters. Here we use the traditional method and assume that $\boldsymbol{\beta}$ has the noninformative prior, $\pi(\boldsymbol{\beta}) \propto 1$. We note that $\pi(\boldsymbol{\beta})$ is an improper prior. It is pointed out in [9] that, for Weibull models $t \sim W(\lambda, \alpha)$, with unknown α and λ, a typical joint prior assumption is to take α and λ to be independent, where α has a gamma distribution denoted by $\alpha \sim G(\kappa_0, \eta_0)$ and λ has a normal distribution. However, in our example given later, we have found that the assumption of gamma prior on λ is more applicable. Hence we assume that λ has a gamma prior distribution represented by $\lambda \sim G(\omega_0, \psi_0)$.

When performing reliability analysis for complex system, sometimes, reliability data of similar systems or historical reliability data for past studies is very helpful to interpret the results of the current study. To introduce the “historical data” in the current study, we consider the power prior for those data.

Suppose we have historical data from a similar previous study, denoted by $D_0 = (n_0, \mathbf{t_0}, \boldsymbol{\upsilon_0}, \mathbf{X_0})$. n_0 is here the sample size of the historical data, $\mathbf{t_0}$ and $\boldsymbol{\upsilon_0}$ are the observed values in the previous study, and $\mathbf{X_0}$ is the $n_0 \times p$ matrix of covariates based on the historical data. The power prior is defined as the likelihood based on historical data D_0, raised to a power a_0. The power a_0 is hence a scalar parameter that controls the influence of the historical data on the current data. Using this model we get

$$\pi(\cdot|D_0, a_0) \propto L(\cdot|D_0)^{a_0} \pi(\cdot|\mathbf{c_0}) \tag{12}$$

where $\pi(\cdot|\mathbf{c_0})$ is the initial prior for historical data and c_0 is a specified hyperparameter for the initial prior (such as α , λ and β mentioned above). Here $a_0 \in [0,1]$, and when $a_0 = 0$, the prior does not depend on the historical data as $\pi(\cdot|D_0, a_0) \propto \pi(\cdot|\mathbf{c_0})$. Hence $a_0 = 0$ corresponds to a prior specification with no incorporation of historical data; while as $a_0 = 1$, $\pi(\cdot|D_0, a_0) \propto L(\cdot|D_0)\pi(\cdot|\mathbf{c_0})$ and (12) corresponds to the update of $\pi(\cdot|\mathbf{c_0})$ using Bayes theorem. When the value of a_0 is between 0 and1, the hierarchical power prior specification is completed by specifying a prior distribution for a_0 . This leads to a joint power prior distribution as shown in (13)

$$\pi(\cdot, a_0|D_0) \propto L(\cdot|D_0)^{a_0} \pi(\cdot|c_0)\pi(a_0|\boldsymbol{\zeta_0}) \tag{13}$$

where $\boldsymbol{\zeta_0}$ is hyperparameter vector for a_0. Common choices for $\pi(a_0|\boldsymbol{\zeta_0})$ is the beta distribution, the truncated gamma distribution, and the truncated normal distribution. These three priors for a_0 all share similar theoretical properties and computational properties. Furthermore, in practice they yield similar results when the hyperparameter is appropriately chosen. For convenience, we will use a beta distribution for $\pi(a_0)$, denoted by $a_0 \sim B(e_0, f_0)$ and thus $\pi(a_0) \propto a_0^{e_0-1}(1-a_0^{f_0-1})$. Further motivation supporting this choice can be found in [10].

3.2 The Posteriors

Once we have settled on a model, the task is now to infer the desired parameters α,λ, and β. For this purpose, we resort to Monte Carlo integration which essentially draws samples from the required distribution, and then forms sample averages to approximate expectations. MCMC draws these samples by running a cleverly constructed Markov chain for a long time. There are many ways of constructing these chains. Perhaps one of the simplest MCMC sampling algorithms found in the Bayesian computational literature is the Gibbs sampler. Literature about MCMC method using Gibbs sampler is too vast to be listed here. Details are referred to [16] and references therein. In this article, the method is used to integrate over the posterior distribution of model parameters given the data, this to make inference for the desired model parameters. Under the model assumptions above, the joint posterior density of $(\boldsymbol{\beta}, \alpha, \lambda, a_0)$ based on current study data set $D = (n, \mathbf{t}, \boldsymbol{\upsilon}, \mathbf{X})$ and historical data set $D_0 = (n_0, \mathbf{t_0}, \boldsymbol{\upsilon_0}, \mathbf{X_0})$ is then given by

$$\begin{aligned}\pi(\boldsymbol{\beta}, \alpha, \lambda, a_0 \mid D, D_0) \propto & \left(\sum_N L(\boldsymbol{\beta}, \alpha, \lambda \mid D)\right) \times \left(\sum_{N_0} L(\boldsymbol{\beta}, \alpha, \lambda \mid D_0)\right)^{a_0} \\ & \times \pi(\boldsymbol{\beta})\pi(\alpha|\kappa_0, \eta_0)\pi(\lambda \mid \omega_0, \psi_0) \times a_0^{e_0-1}(1-a_0^{f_0-1})\end{aligned} \tag{14}$$

where,

$$\begin{aligned}\left(\sum_{N_0} L(\boldsymbol{\beta}, \alpha, \lambda \mid D_0)\right)^{a_0} & \propto \prod_{i=1}^{n_0} (\theta_{0i} f(t_{0i} \mid \alpha, \lambda)^{a_0 \upsilon_{0i}} \exp(-a_0\theta_{0i}(1 - R(t_{0i} \mid \alpha, \lambda))) \\ & = \sum_{N_0} \prod_{i=1}^{n_0} \left[(\theta_{0i} f(t_{0i} \mid \alpha, \lambda)^{a_0 \upsilon_{0i}} \frac{1}{N_{0i}!} \exp(-a_0\theta_{0i})(a_0\theta_{0i} R(t_{0i} \mid \alpha, \lambda))^{N_{0i}} \right]\end{aligned} \tag{15}$$

and $\theta_{0i} = \mathbf{X_0'}\boldsymbol{\beta}$. Denote $D_{obs} = (D, D_0)$, then the joint posterior of $(\boldsymbol{\beta}, \alpha, \lambda, a_0, N, N_0)$ can be written as:

$$\begin{aligned}\pi(\boldsymbol{\beta},\alpha,\lambda,a_0,N,N_0 \mid D_{obs}) \propto & \prod_{i=1}^{n} R(t_i \mid \alpha,\lambda)^{N_i-\upsilon_i}(N_i f(t_i \mid \alpha,\lambda))^{\upsilon_i} \\ & \times \exp[\sum_{i=1}^{n}(N_i \mathbf{x}_i^{'}\boldsymbol{\beta} - \log(N_i!) - \exp(\mathbf{x}_i^{'}\boldsymbol{\beta}))] \\ & \times \prod_{i=1}^{n_0}(\exp(\mathbf{x}_{0i}^{'}\boldsymbol{\beta}) f(t_{0i} \mid \alpha,\lambda))^{a_0\upsilon_{0i}}(R(t_{0i} \mid \alpha,\lambda))^{N_{0i}} \\ & \times \exp[\sum_{i=1}^{n_0}(N_{0i}(\log(a_0) + \mathbf{x}_{0i}^{'}\boldsymbol{\beta} - \log(N_{0i}!) - a_0\exp(\mathbf{x}_{0i}^{'}\boldsymbol{\beta}))] \\ & \times \pi(\boldsymbol{\beta})\pi(\alpha|\kappa_0,\eta_0)\pi(\lambda \mid \omega_0,\psi_0) \times a_0^{e_0-1}(1-a_0^{f_0-1})\end{aligned} \tag{16}$$

Samples from this distribution can now be used for inference regarding the desired parameters. As mentioned we use Gibbs sampling to achieve samples. This strategy does sampling in one dimension at a time. For this purpose, the distributions of each parameter conditioned on the others are required. These can be written as

- $\pi(\boldsymbol{\beta}_m \mid \boldsymbol{\beta}^{(-m)},\alpha,\lambda,a_0,N,N_0,D_{obs}) \propto \pi(\boldsymbol{\beta},\alpha,\lambda,a_0,N,N_0 \mid D_{obs}) \times 1$
- $\pi(\alpha_m \mid \alpha^{(-m)},\lambda,\boldsymbol{\beta},a_0,N,N_0,D_{obs}) \propto \pi(\boldsymbol{\beta},\alpha,\lambda,a_0,N,N_0 \mid D_{obs}) \times \alpha_j^{\kappa_0-1} \times \exp(-\eta_0\alpha_j)$
- $\pi(\lambda_m \mid \lambda^{(-m)},\alpha,\boldsymbol{\beta},a_0,N,N_0,D_{obs}) \propto \pi(\boldsymbol{\beta},\alpha,\lambda,a_0,N,N_0 \mid D_{obs}) \times \lambda^{\omega_0-1}(1-\lambda^{\psi_0-1})$
- $\pi(a_{0m} \mid \alpha_0^{(-m)},\lambda,\boldsymbol{\beta},\alpha,N,N_0,D_{obs}) \propto \pi(\boldsymbol{\beta},\alpha,\lambda,a_0,N,N_0 \mid D_{obs}) \times a_0^{e_0-1}(1-a_0^{f_0-1})$

4. Illustrative Example

4.1 The Data

We now consider the example data discussed in [17]. The reliability data treats lifetimes of pressure vessels for a Space Shuttle at four different fiber stresses (29.7MPa, 27.6 MPa, 25.5 MPa, 23.4 MPa; MPa means MegaPascals) and for eight spools in a random truncated test. To exemplify the use of the power prior, randomly choose 81 of the data as reliability data in the current study as shown in Table1, whereas the other 27 data are chosen as historical data as shown in Table 2. Suppose that $\mathbf{x}_i^{'}\boldsymbol{\beta} = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i}$ where β_0, x_1, x_2, β_1, β_2 denote the intercept, the four stress level, the eight different spools, and the regression coefficients of x_1 and x_2, respectively. We note that in [17], the influences of the spools are viewed as an unknown random effect. Let us use four settings for the power prior $a_0 = 0$, $a_0 \sim B(200,1)$, $a_0 \sim B(400,1)$, $a_0 = 1$, separately. The means for $E(a_0|D_{obs})$ therefore equals 0, 0.14, 0.29, 1 respectively. Furthermore, let us use $\lambda \sim G(1,0.01)$ and $\alpha \sim G(1,0.01)$ which are common in Bayesian analysis. Let the failure time t_i be the observed data if failure happened during the test. For the truncated data, we only know that it exceeds the observable time L_i, which are denoted by asterisk.

Table 1: Failure Time in Random Truncated Test for Current Study

Stress (MPa)	Spool	t_i / L_i (hour)	Stress (MPa)	Spool	t_i / L_i (hour)	Stress (MPa)	Spool	t_i / L_i (hour)	Stress (MPa)	Spool	t_i / L_i (hour)
29.7	2	2.2	29.7	4	254.1	27.6	1	930.4	25.5	1	14032
29.7	7	4	29.7	1	444.4	27.6	6	1254.9	25.5	4	29808
29.7	7	4	29.7	8	590.4	27.6	4	1275.6	25.5	1	31008
29.7	7	6.1	29.7	1	755.2	27.6	1	1755.5	23.4	7	5376
29.7	6	6.7	29.7	1	952.2	27.6	8	2046.2	23.4	6	7320
29.7	7	7.9	29.7	1	1108.2	27.6	4	6177.5	23.4	3	8616
29.7	2	8.5	29.7	4	1569.3	25.5	7	503.6	23.4	2	14400
29.7	2	9.1	29.7	4	1750.6	25.5	3	1087.7	23.4	6	16104
29.7	2	10.2	29.7	4	1802.1	25.5	2	1134.3	23.4	5	20231
29.7	5	13.3	27.6	3	24.3	25.5	2	1920.1	23.4	5	35880
29.7	7	14	27.6	3	69.8	25.5	2	2383	23.4	1	41000*
29.7	3	14.6	27.6	2	71.2	25.5	3	2442.5	23.4	1	41000*
29.7	3	18.7	27.6	2	199.1	25.5	2	3708.9	23.4	1	41000*
29.7	2	22.1	27.6	2	403.7	25.5	8	4908.9	23.4	4	41000*
29.7	7	45.9	27.6	2	432.2	25.5	2	5556	23.4	4	41000*
29.7	7	61.2	27.6	2	514.1	25.5	8	7332	23.4	4	41000*
29.7	5	87.5	27.6	6	514.2	25.5	8	7918.7	23.4	8	41000*
29.7	8	98.2	27.6	6	541.6	25.5	6	7996	23.4	8	41000*
29.7	2	111.4	27.6	8	554.2	25.5	8	9973			
29.7	6	144	27.6	1	664.5	25.5	1	11487			
29.7	2	158.7	27.6	2	694.1	25.5	5	11727			

Table 2: Failure Time in Random Truncated Test for Historical Study

Stress (MPa)	Spool	t_i / L_i (hour)	Stress (MPa)	Spool	t_i / L_i (hour)	Stress (MPa)	Spool	t_i / L_i (hour)	Stress (MPa)	Spool	t_i / L_i (hour)
29.7	7	4.6	29.7	8	638.2	27.6	4	1536.8	23.4	7	4000
29.7	5	8.3	29.7	4	1148.5	25.5	6	225.2	23.4	5	9120
29.7	3	12.5	27.6	3	19.1	25.5	2	1824.3	23.4	6	20233
29.7	6	15	27.6	3	136	25.5	8	2974.6	23.4	1	41000*
29.7	2	55.4	27.6	1	453.4	25.5	6	6271.1	23.4	4	41000*
29.7	3	101	27.6	2	544.9	25.5	8	9240.3	23.4	8	41000*
29.7	5	243.9	27.6	4	876.7	25.5	4	13501	23.4		

4.2 Analysis

Using a burn-in of 10000 samples, and basing our analyses on 290000 Gibbs samples, we get the following posterior summaries using the different a_0 priors ($E(a_0|D_{obs})=0$ 、0.14、0.29、1), as shown in Table 3. The results include posterior mean, posterior standard deviation, MC errors, and 95% highest posterior density (HPD) intervals for the included model parameters with different power prior a_0. Table3 shows that, the MC errors are small <0.06) that improves the effective of the model. Also, we find that the 95% HPD interval for β_2 includes 0, which means that the influences given by different spools are in fact uncertain. This conclusion is consistent with the one given in [13], in which the influences given by different spools are viewed as an unknown random effect. Additionally, posterior mean for regression coefficients (β_0, β_1, and β_2) do not change a great deal if a_0 changes form 0 to 1. However, the 95% HPD intervals become shorter, which means that we trust for "historical data" more, and that is because the "historical data" we used here are essentially coming from "current study". On the other hand, if the value for some coefficient changes a lot, it indicates that the coefficient is a potentially important factor and need to be studied further (see [18]).

Table 3: Posterior Summaries for the Data

$E(a_0)$	Parameter	Mean	Stand. Deviation	MC Error	95% HPD Interval
0	β_0	-3.072	1.483	0.05717	(-6.233, -0.1835)
	β_1	0.1126	0.05307	0.002049	(0.00863, 0.2247)
	β_2	0.007262	0.04824	4.538E-4	(-0.08777, 0.1006)
	λ	0.01001	0.009936	3.046E-5	(2.556E-4, 0.03669)
	α	0.106	0.008689	1.634E-5	(0.08911, 0.1231)
0.14	β_0	-3.027	1.318	0.05101	(-5.652, -0.5075)
	β_1	0.1108	0.04696	0.001815	(0.02078, 0.2042)
	β_2	0.009525	0.04394	4.666E-4	(-0.07708, 0.09523)
	λ	0.008063	0.008022	2.628E-5	(1.982E-4, 0.02964)
	α	0.1066	0.007791	1.529E-5	(0.09149, 0.1219)
0.29	β_0	-2.823	1.305	0.05075	(-5.556, -0.2461)
	β_1	0.1034	0.04658	0.00181	(0.0114, 0.2007)
	β_2	0.00936	0.043	4.746E-4	(-0.0751, 0.0939)
	λ	0.007623	0.007594	2.447E-5	(1.923E-4, 0.02813)
	α	0.1067	0.007608	1.419E-5	(0.09192, 0.1217)
1	β_0	-3.016	1.221	0.04653	(-5.401, -0.7153)
	β_1	0.1104	0.04365	0.001663	(0.02799, 0.1963)
	β_2	0.009645	0.04291	4.692E-4	(-0.07445, 0.09437)
	λ	0.007621	0.007599	2.395E-5	(1.91E-4, 0.02802)
	α	0.1067	0.007624	1.526E-5	(0.09191, 0.1218)

In conclusion, the cure fraction with $a_0 = 1$ can be given by $\pi = \exp[-\exp(-3.016 + 0.1104x_1 + 0.009645x_2)]$, (compared with $\pi = \exp(-\theta)$), and the reliability can be given by:

$$R_p(t) = \exp[-\exp(-3.016 + 0.1104x_1 + 0.009645x_2) \times (1 - \exp(0.007621)t^{0.1067})]$$

Table 4: Cure Fraction π

π	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
29.7	0.2726	0.2685	0.2647	0.261	0.2576	0.2544	0.2515	0.2489
27.6	0.3538	0.35	0.3463	0.3426	——	0.3353	——	0.3283
25.5	0.4363	0.433	0.4295	0.426	0.4223	0.4185	0.4147	0.4108
23.4	0.5146	0.5117	0.5086	0.5054	0.5019	0.4982	0.4944	0.4904

"—"denotes the results cannot be given due to original incomplete data

Table 4 gives the associated values for the cure fraction π for the system. We mark these values in Fig2. From Fig.2 we can see clearly that, due to the differences between different spools (random effect referred in [17]), the points do not obey superposition. However, the cure fraction with the same stress seems much closer. The cure fraction ($\pi \neq 0$) shows the model's rationality once more. Based on the trend curve in Fig.2, one can obviously find that, when stresses become larger, π becomes smaller, which means the ratio for subsystems which may influence the life of the system become larger (change form 25% to 51%). That conclusion is consistent with the situation in practice and is not possible based on frequents or Bayesian analysis of the current data alone.

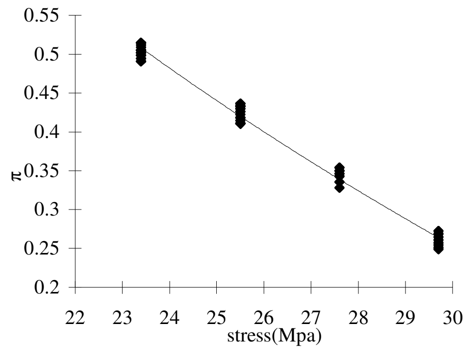

Figure 2: Plot for Cure Fraction π

5. Conclusion

In this article, we have introduced the cure rate model to perform reliability analysis. Based on these ideas, we have explored on an alternative model in which its Bayesian formulation is used for reliability analysis in complex system. By using a regression model for the cure fraction as well as application of the power prior, we have provided a new method to perform reliability analysis in complex system; by which, some conclusions cannot be obtained by traditional methods. The results have also illustrated that, the model can improve robustness, including making estimates for regression coefficients more accurately. As a consequence, these also make inference for the lifetime of the complex system more accurate. Because the method needs fewer hypotheses, this predominance is more distinctly when there is not enough prior information and there exist truncated data in the model. We have used MCMC methods (Gibbs sampler) to integrate over the high-dimensional probability distribution in order to make inference for model parameters and to make predictions. A limiting factor in our analysis is that the t_i s' are assumed i.i.d, in future works we will address this issue.

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