

A Cost Model for Repairable System Considering Multi-failure type over Finite Time Horizon

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(Received on September 6, 2009, revised on August 9, 2010)

Abstract: In general, downtime of a system can be attributed due to multiple failure categories and repair costs for each failure categories can be different. Many of these failure types are repaired to a state which can be called as bad as old and such repair actions are termed as “minimal repair”. Many system or components are replaced after a certain number of such minimal repair actions. In this study, we intend to prove that if the system failure process can be described by NHPP (Non Homogenous Poisson Process), then each failure category can also be modelled by NHPP. Based on this, a cost model is developed by using the decomposition of the NHPP and renewal theory. Using the cost model, a model is developed to obtain the optimal number of minimum repair action every failure category. Since it is not possible to find any analytical solution, solution to the renewal function, an approximate approach is introduced to obtain numerical solution. Finally, a numerical example is presented to demonstrate the method

Keywords: *Multiple failure types, NHPP, renewal function, finite time horizon, numerical solution.*

1. Introduction

The failure of complex system usually comprises various failure modes, failure causes, or failure mechanisms. Such failure usually originates from its different subsystems or components. The repair cost against each failure type can be different. Take the example of Load-Haul-Dump machine from mining industry. Major failures of the machine are due to engine, hydraulic, brake, or tyres failure [1]. On repairing engine, hydraulic, and so on, their cost varies from each other.

In literature survey [2-7], it is found that most replacement models base on the reward renewal process and consider infinite time horizon. Advantage of those models is analytical solution can be obtained. While in practice, the time horizon of interest could be short, so these long run models may not be accurate. Jack presented a comparison between finite time and infinite time horizon [3], which reveals the replacement policy based on finite time horizon can reduce unit time cost more than 2.92% in the example he presented. It is demonstrated the cost considering finite time horizon is better than infinite

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time horizon for some cases. Recently, there are more replacement optimization models considering finite time horizon (*e.g.*, Castro and Alfa [8], Hartman and Murphy [9]). However, in general, most of those models assume the repair cost against all types of failure is identical or considers only one type of minimal repair. It is unrealistic in some cases. The paper extends the number of failure types from one to more than one, and accordingly the repair cost against each type of failure need not be identical. A cost model is developed to determine the optimum maximum minimal repairs for each type of failure.

In the remaining sections of this paper, the proof on decompositions of NHPP also follow NHPP is firstly presented. Then it presents a cost models based on renewal theory. Based the cost model, a replacement optimization model is developed. Then an approximate approach is introduced to obtain numerical solution. Finally, a numerical example is presented.

2. Problem Description

The paper assumes failure process of the system follows NHPP [10]. NHPP is the common failure process, which is most acceptable in reliability analysis for repairable system [11]. As mentioned in Section 1, system failures comprise of multiple types of failure. Conditioning on system failure has occurred, assume the probability of type i failure is $P_i(t)$. Therefore,

$$P_1(t) + P_2(t) + \dots + P_n(t) = 1 \quad (1)$$

Assume the system will be replaced after a predetermined number of minimal repairs. Let (n_1, n_2, \dots, n_n) denotes the maximum number of minimal repairs the system can tolerate for each failure type. When any type of failures reaches its n_i , the system will be replaced.

Sheu and Griffith [7] have proved two decompositions (Special case of Equation (1) where $n = 2$) of NHPP also follows NHPP, *i.e.* at each instantaneous time, when event (Main event) occurs, their two sub events also follows NHPP. In this paper, it is proved in section 3.1 that for $n > 1$, decommissions of NHPP also follows NHPP, *i.e.* each individual failure type also follows NHPP with failure rate:

$$\lambda_i(t) = P_i(t)\lambda(t) \quad (i = 1, 2, 3, \dots, n), \quad (2)$$

where $\lambda(t)$ denotes system failure rate, n denotes number of failure types.

In this problem, the replacement process follows renewal process. Within each replacement cycle, it is a superposition of several NHPP. Each type of failure is competing to reach replacement, which is equivalent to the competition to reach failure in ordinary competing risks models. The cost model developed in Section 4 bases on this renewal process.

3. System Failure Rate and its Decomposition

3.1 Probability of Failure Occurrence

Assume a failure occurs during time interval $[0, t]$. Let X denotes the time when failure occurred. Then the probability of the failure occurrence conditioning on one failure occurs in $[0, t]$ is:

$$P\{X < s | N(t) = 1\} = \frac{P\{X < s, N(t) = 1\}}{N(t) = 1} \tag{3}$$

Equation (3) can be rewritten to

$$P\{X < s | N(t) = 1\} = \frac{P\{N(s) = 1, N(s, t) = 0\}}{N(t) = 1} \tag{4}$$

Due to independent increment of *NHPP*[12,13], the disjoint interval are independent, thus

$$P\{N(s) = 1, N(s, t) = 0\} = P\{N(s) = 1\}P\{N(s, t) = 0\} \tag{5}$$

Hence,

$$P\{X < s | N(t) = 1\} = \frac{\Lambda(s)e^{-\Lambda(s)} \cdot \Lambda(s, t)e^{-\Lambda(s, t)}}{\Lambda(t)e^{-\Lambda(t)}} = \frac{\Lambda(s)}{\Lambda(t)} \tag{6}$$

where $\Lambda(t)$ denotes cumulative failures, which equals to $\int_0^t \lambda(s) ds$. Differentiating

Equation(4), the probability density of system failure is therefore obtained as

$$f(s) = \frac{\lambda(s)}{\Lambda(t)} \tag{7}$$

Hence, the probability of *i*th type of failure occurs during interval [0,t] is

$$Q_i(t) = \int_0^t f(s)P_i(t)ds = \frac{\int_0^t \lambda(s)P_i(s)ds}{\Lambda(t)} \tag{8}$$

3.2 Decomposition of NHPP

As mentioned in Section 2, the system failure comprises of *n* types of failure. The probability of *i*th failure varies with time $P_i(t)$ (*i*=1,2,...,n). Then the process of the *i*th failure also follows NHPP and each sub failure process is independent from each other with failure rate:

$$\lambda_i(t) = P_i(t)\lambda(t) \quad (i=1,2,3,\dots,n) \tag{9}$$

Similar to the proof in Reference [12] for two types of sub events (Failure types), there are *n* types of events (Failure types) in this paper. $N_i(t)$ denotes number of type *i* failure experienced within time *t*. The joint distribution of $N_1(t), N_2(t), \dots, N_n(t)$ is

$$P\{N_1(t) = n_1, N_2(t) = n_2, \dots, N_n(t) = n_n\} \tag{10}$$

where $n_1 + n_2 + \dots + n_n = n$, Equation (10) equals to

$$P\{N_1(t) = n_1, \dots, N_n(t) = n_n | N(t) = n\}P\{N(t) = n\} \tag{11}$$

where

$$P\{N(t) = n\} = \frac{\Lambda(t)^n}{n!} e^{-\Lambda(t)} \tag{12}$$

$P\{N_1(t) = n_1, \dots, N_n(t) = n_n | N(t) = n\}$ implies: given *n* system failure occurred, the number of type *1* failure in *n* is *n*₁, type 2 is *n*₂,...It essentially follows multinomial distribution.

Therefore,

$$P\{N_1(t) = n_1, \dots, N_n(t) = n_n | N(t) = n\} = \frac{n!}{n_1! n_2! n_3! \dots n_n!} Q_1(t)^{n_1} Q_2(t)^{n_2} \dots Q_n(t)^{n_n} \quad (13)$$

where $P_1 + \dots + P_i + \dots + P_n = 1$ and $Q_i(t)$ is defined in Equation (8).

Substituting (11) by (12)(13), Equation(11) can be rewritten to:

$$P\{N_1(t) = n_1, N_2(t) = n_2, \dots, N_n(t) = n_n\} = \frac{\left[\int_0^t \lambda(s) P_1(s) ds\right]^{n_1} e^{-[P_1(t)\Lambda(t)]} \left[\int_0^t \lambda(s) P_2(s) ds\right]^{n_2} e^{-[P_2(t)\Lambda(t)]} \dots \left[\int_0^t \lambda(s) P_n(s) ds\right]^{n_n} e^{-[P_n(t)\Lambda(t)]}}{n_1! n_2! \dots n_n!} \quad (14)$$

In the right part of Equation(14),

$$P\{N_i(t) = n_i\} = \frac{\left[\int_0^t \lambda(s) P_i(s) ds\right]^{n_i}}{n_i!} e^{-\left[\int_0^t \lambda(s) P_i(s) ds\right]} \quad (15)$$

$$\text{Hence, } P\{N_1(t) = n_1\} P\{N_2(t) = n_2\} \dots P\{N_n(t) = n_n\} = P\{N_1(t) = n_1, N_2(t) = n_2, \dots, N_n(t) = n_n\} \quad (16)$$

i.e., each failure type is independent from another and the *i*th type of failure also follows *NHPP* with failure rate $\lambda(s)P_i(s)$. Equation (9) is hence proved.

3 Cost Model Development

3.1 Interarrival Time Distribution

As mentioned in Section 2, failure rates of each failure type are $p_1(t)\lambda(t)$, $p_i(t)\lambda(t)$, $p_n(t)\lambda(t)$, respectively. Let n_i denote the maximum number of repairs for type *i* failure. Then the probability of type *i* failure leading to replacement is:

$$F_i(t) = 1 - R_i = 1 - P(N_i(t) < n_i) = 1 - \sum_{k=0}^{n_i-1} \frac{\exp\left\{-\int_0^t p_i(t)\lambda(t)dt\right\} \left[\int_0^t p_i(t)\lambda(t)dt\right]^k}{k!} \quad (17)$$

Due to independence of each failure type's process, these failure types are competing to reach replacement. Then interarrival time distribution is:

$$F(t) = 1 - R_1(t)R_2(t) \dots R_j(t) \dots R_n(t) \quad (18)$$

3.2 Expected Cost

The expectation of total cost to time *t* can be represented as [2]:

$$E\{C(t) | Z_1 = u\} = \begin{cases} \alpha(u) + C(t-u), & u \leq t \\ \beta(t), & u > t \end{cases} \quad (19)$$

where Z_1 denotes the time point of first replacement occurrence within time *t*. $\alpha(u)$ denotes the cost when replacement occurs at time point *u*. $C(t-u)$ is the recursive formulate of renewal process. $\beta(t)$ denotes the cost conditioned on no replacement

occurrence within time t . Take the integral of Equation (19). Then the total cost within time t can be rewritten to:

$$\begin{aligned}
 C(t) &= \int_0^{\infty} E\{C(t)|Z_1 = u\}dF(u) \\
 &= \beta(t)R(t) + \int_0^t [\alpha(u) + C(t-u)]dF(u) \\
 &= \beta(t)R(t) + \int_0^t \alpha(u)dF(u) + \int_0^t C(t-u)dF(u) \\
 &= A(t) + \int_0^t C(t-u)dF(u)
 \end{aligned} \tag{20}$$

Let $(m_1, m_2, \dots, m_i, \dots)$ denote the number of minimal repairs that the system has experience since the latest replacement. $(c_1, c_2, \dots, c_i, \dots)$ denotes the minimal repair cost for each failure type. n_i denotes the maximum number of minimal repairs (called threshold in this paper) for type i failure. When no replacement occurred, i.e. $(m_1 < n_1, m_2 < n_2, \dots, m_i < n_i, \dots)$, the cost will be

$$\beta(t) = \frac{\sum_{m_1=0}^{n_1-1} \sum_{m_2=0}^{n_2-1} \dots \sum_{m_i=0}^{n_i-1} (m_1 \times c_1 + m_2 \times c_2 + \dots) \cdot Po_1(t, m_1) \cdot Po_2(t, m_2) \dots}{R(t)} \tag{21}$$

where,

$$Po_i(t, m_i) = \frac{\exp\left\{-\int_0^t p_i(t)\lambda(t)dt\right\} \int_0^t p_i(t)\lambda(t)dt^{m_i}}{m_i!}$$

When system replacement occurred, it implies that at least one failure type reached its threshold. Therefore, the cost will be:

$$\begin{aligned}
 &\int_0^t [\alpha(u) + C(t-u)]dF(u) \\
 &= \int_0^t [c_1 \times (n_1 - 1) + c_2 \times m_2 \dots] d((1 - R_1)Po_2(u, m_2) \dots Po_i(u, m_i) \dots) \\
 &+ \int_0^t [c_1 \times m_1 + c_2 \times (n_2 - 1) + \dots c_i \times m_i \dots] d((1 - R_1)Po_2(u, m_2) \dots Po_i(u, m_i) \dots) \\
 &\dots \\
 &+ \int_0^t [c_1 \times (n_1 - 1) + c_2 \times (n_2 - 1) + \dots c_i \times (n_i - 1) \dots] d((1 - R_1)(1 - R_2) \dots (1 - R_i) \dots)
 \end{aligned} \tag{22}$$

where $(1 - R_1)Po_2(u, m_2) \dots Po_i(u, m_i) \dots$ denotes the probability that such scenario will happen: type 1 failure reached its threshold, type 2 failure reaches $m_2 < n_2$, ... type i reaches $m_i < n_i$... And $c_1 \times (n_1 - 1) + c_2 \times m_2 \dots$ denotes its corresponding cost. Equation(21) enumerates all combinations of $(m_1 \leq n_1, m_2 \leq n_2, \dots, m_i \leq n_i, \dots)$ excluding $(m_1 < n_1, m_2 < n_2, \dots, m_i < n_i, \dots)$.

Substitute Equation (21) (22) to Equation (20). Following Equation is obtained.

$$A(t) = c_1 \sum_{m_1=0}^{n_1-1} m_1 \times Po_1(t, m_1) + c_1(n_1 - 1) \left[1 - \sum_{i=0}^{n_1-1} Po_1(t, m_1) \right] +$$

$$c_2 \sum_{m_2=0}^{n_2-1} j \times P_{O_2}(t, m_2) + c_2(n_2 - 1) \left[1 - \sum_{m_2=0}^{n_2-1} P_{O_1}(t, m_2) \right] + \dots + c_r F(t) \quad (23)$$

Let $F^{(i)}(t) = 1 - \sum_{q=0}^i P_{O_i}(t, q)$. Equation (23) hence derivates to:

$$A(t) = c_1 \sum_{j=1}^{n_1-1} F^{(j)}(t) + \dots + c_m \sum_{j=1}^{n_m-1} F^{(j)}(t) + c_r F(t) \quad (24)$$

Therefore, the total cost model can be rewritten as:

$$C(t) = c_1 \sum_{i=1}^{n_1-1} F^{(i)}(t) + \dots + c_m \sum_{p=1}^{n_m-1} F^{(i)}(t) + c_r F(t) + \int_0^t C(t-u) dF(u) \quad (25)$$

3.3 Replacement Optimization Model

The total cost $C(T)$ within time horizon T can be obtained from Equation (25). The cost per time unit is

$$L(T) = \frac{C(T)}{T} \quad (26)$$

The replacement optimization model is to obtain the optimal number of minimal repairs for each type of failure. Take (n_1, n_2, \dots) , which denotes the maximum number of minimal repairs for each failure type, as decision variables. Take minimum cost per time unit as objective function. Rewrite the Equation (26) in a nonlinear programming format. Then,

$$\begin{aligned} & \text{Min } \frac{C(T; n_1, \dots, n_i, \dots, n_n)}{T} \\ & \text{s.t. } n_i = 1, 2, 3, \dots; i = 1, 2, \dots, n \end{aligned} \quad (27)$$

where n_i is integer. The best solution to equation (27) is the optimal number of minimal repairs. Matlab optimization toolbox can be used to obtain the best solution.

3.4 Approximate to Renewal Function

It is hard to obtain analytical solution for renewal function of (25). Unless the interarrival function follows exponential distribution, it is not possible to obtain renewal function analytically for most other distributions, including the Weibull Distribution [14]. Nevertheless, there are plenty of approximate methods that can be used. This paper uses an approach developed by Tortorella [15]. The cost model is approximated by following formulation, which is adapted from formula (2.5) of [15].

$$C_m(i) = \frac{2A_i}{2 - F_i} + \sum_{k=1}^{i-1} A_{i-k} \frac{F_{k+1} - F_{k-1}}{2 - F_1} \quad (28)$$

This approach uses Trapezoid rule to solve integral problem. In the formula, i denotes the time of i th step, which equals $\frac{i}{N} \cdot t$. $\frac{t}{N}$ denotes the step size to control the accuracy of the approximation solution. F_i and A_i can be calculated by Equation (18) and (24), respectively.

4. Numerical Example

To demonstrate the approach developed in this paper, a simple example is presented. The example implements the approach using Matlab. Assume the $P_i(t)$ is constant. Given failure of the system follows NHPP with shape and scale parameter $\alpha = 2, \beta = 1$, and the system suffers two failure modes.

Suppose the repair costs for each failure type are $c_1 = 10; c_2 = 20$; replacement cost $c_r = 100$; $p_1(t) = 0.3; p_2(t) = 0.7$. Figure 1 and Figure 2 illustrate the relationship between per time unit cost and (n_1, n_2) when time horizon $T = 4$.

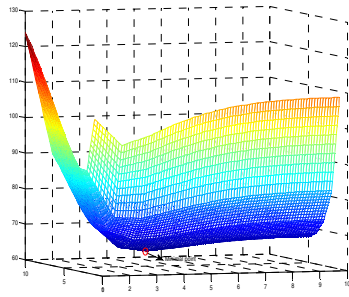


Figure 1: Minimal Cost

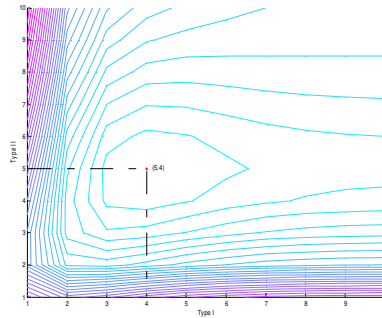


Figure 2: Contour Plot

Figure 1 shows the minimal cost per time unit exists. Figure 2 shows the optimum minimal repair number of type 1 and type 2 failure is (4,5), i.e., after 4 type I minimal repairs or 5 type II, the system should be replaced.

To find how the replacement cost influences the optimum number of minimal repairs, a comparison is performed given various replacement cost against time horizon. Supposes $c_1 = 10; c_2 = 20; p_1(t) = 0.3; p_2(t) = 0.7$. Table 1 list out all optimum a solutions. In general, it is concluded that with replacement cost increasing, the optimum number of minimal repairs increasing accordingly. When $c_r = 200$, the number of repairs is unlimited for time horizon 4, which means it is never needed to be replaced.

Table 1: Replacement Opportunity against Replacement Cost

$T \backslash P_i$	$c_r = 100$	$c_r = 200$	$c_r = 400$	$c_r = 800$
4	(4,5)	(*,*)	(*,*)	(*,*)
8	(3,4)	(10,16)	(10,17)	(12,18)
12	(3,4)	(9,14)	(9,15)	(10,16)

As mentioned in Section 1, it is reasonable to consider finite time horizon other than its infinite counterpart. Figure 3 shows the cost per time unit is increasing with time horizon.

The cost per time unit approaches to constant when time approaching to infinite. Therefore when time horizon is large enough, the time horizon can be considered as infinite. Figure 3 shows that the cost per unit time varies greatly with time horizon, especially when time horizon is short, it highlights the advantage of finite time horizon model.

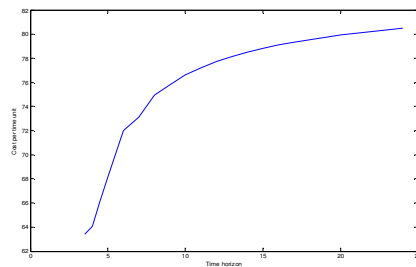


Figure 3: Cost per Time Unit

6 Conclusion

The paper generalized some existing models which consider only one minimal repair and finite time horizon is considered. The paper also developed a cost model and presented an approach to obtain numerical solution to approximate renewal function. To obtain optimum number of minimal repairs, a nonlinear programming formulation is developed. The paper doesn't discuss infinite time horizon model. While as shown in the numerical example, when time horizon is large enough, the cost per time unit approaches constant, which implies the time horizon can be considered as infinite.

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